Unitary Representations of the 2-Dimensional Euclidean Group in the Heisenberg Algebra

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Abstract: $E(2)$ is studied as the automorphism group of the Heisenberg algebra H. The basis in the Hilbert space K of functions on H on which the unitary irreducible representations of the group are realized is explicitly constructed. The addition theorem for the Kummer functions is derived.

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1. Introduction

Investigating the properties of manifolds by means of the symmetries they admit has a long history. Non-commutative geometries have become the subject of similar studies in recent decades. For example there exists an extensive literature on the q-deformed groups $E_q(2)$ and $SU_q(2)$ which are the automorphism groups of the quantum plane $zz^* = qz^*z$ and the quantum sphere respectively [1]. Using group theoretical methods the invariant distance and the Green functions have also been written in these deformed spaces [2].

The purpose of the present work is to analyze yet another non-commutative space $[z, z^*] = \sigma$ (i. e. the space generated by the Heisenberg algebra) by means of its automorphism group $E(2)$.

In Section 2 we define $E(2)$ in the Heisenberg algebra H and construct the unitary representations of the group in the Hilbert space X where H is realized.

In Section 3 we give the unitary irreducible representations of $E(2)$ in the Hilbert space K of the square integrable functions on H and construct the basis in K where the irreducible representations of the group are realized. The basis are found to be written in terms of the Kummer functions. Commutative limit as $\sigma \to 0$ is also discussed.

Section 4 is devoted to the addition theorem for the Kummer functions. This theorem provides a group theoretical interpretation for the already existing identities involving the Kummer and Bessel functions. It may also lead to new identities.

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2. $E(2)$ as the automorphism group of the Heisenberg algebra

The one dimensional Heisenberg algebra H is the 3-dimensional vector space with the basis elements $\{z, z^*, 1\}$ and the bilinear antisymmetric product

$$[z, z^*] = 1. \tag{1}$$

The $*$-representation of H in the suitable dense subspace of the Hilbert space X with the complete orthonormal basis $\{\mid n\rangle\}$, $n = 0, 1, 2, ...$ is given by

$$z \mid n\rangle = \sqrt{n} \mid n-1\rangle, \quad z^* \mid n\rangle = \sqrt{n+1} \mid n+1\rangle. \tag{2}$$

Let us represent the Euclidean group $E(2)$ in the vector space H

$$g \begin{pmatrix} z \\ z^* \\ 1 \end{pmatrix} = \begin{pmatrix} e^{i\phi} & 0 & re^{i\psi} \\ 0 & e^{-i\phi} & re^{-i\psi} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ z^* \\ 1 \end{pmatrix}. \tag{3}$$

Since these transformations preserves the commutation relation

$$[gz, gz^*] = [z, z^*] \tag{4}$$

we conclude that

$$gz = U(g)zU^{-1}(g), \quad gz^* = U(g)z^*U^{-1}(g) \tag{5}$$

where $U(g)$ is the unitary representation of $E(2)$ in X:

$$U(g_1)U(g_2) = U(g_1g_2), \quad U^*(g) = U^{-1}(g) = U(g^{-1}). \tag{6}$$

Simple verification shows that the normalized state

$$\mid 0\rangle' = e^{-\frac{r^2}{2}} e^{re^{i(\psi-\phi)}z^*} \mid 0\rangle \tag{7}$$

satisfies the condition

$$gz \mid 0\rangle' = 0. \tag{8}$$

In the new orthonormal basis

$$\mid n\rangle' = \frac{(gz^*)^n}{\sqrt{n!}} \mid 0\rangle' \tag{9}$$

we have

$$gz \mid n\rangle' = \sqrt{n} \mid n-1\rangle', \quad gz^* \mid n\rangle' = \sqrt{n+1} \mid n+1\rangle'. \tag{10}$$

The unitary operator $U(g)$ defines the transition between two complete orthonormal basis $\{\mid n\rangle\}$ and $\{\mid n\rangle'\}$

$$\mid n\rangle' = U(g) \mid n\rangle, \quad \mid n\rangle = U(g^{-1}) \mid n\rangle'. \tag{11}$$

Matrix elements of $U(g)$ in the basis $\{\mid n\rangle\}$ reads

$$\langle m \mid U(g) \mid n\rangle = \frac{e^{-\frac{r^2}{2}}}{\sqrt{n!}} \langle m \mid (gz^*)^n e^{-re^{i(\psi-\phi)}z^*} \mid 0\rangle \tag{12}$$

which after some simple algebra can be expressed in terms of the degenerate hypergeometric functions as

$$\langle m \mid U(g) \mid n\rangle = (-)^m e^{i(m-n)\psi - in\phi} \frac{r^{n+m} e^{-\frac{r^2}{2}}}{\sqrt{n!m!}} {}_2F_0(-m, -n; -\frac{1}{r^2}). \tag{13}$$

Using the relations [5]

$${}_2F_0(-m, -n; -\frac{1}{r^2}) = \frac{n!}{(n-m)!}(-\frac{1}{r^2})^m \Phi(-m, 1+n-m; r^2), \quad n \geq m, \tag{14}$$

$${}_2F_0(-m, -n; -\frac{1}{r^2}) = \frac{m!}{(m-n)!}(-\frac{1}{r^2})^n \Phi(-n, 1+m-n; r^2), \quad m \geq n \tag{15}$$

we can also express the matrix elements in terms of the Kummer function Φ.

3. Unitary representations of $E(2)$ in the space of functions on H

Let K_0 be set of finite sums

$$F = \sum (f_n(\zeta) z^n + z^{*n} f_{-n}(\zeta)). \tag{16}$$

Here $f_n(\zeta)$ are functions of $\zeta = z^* z$ with finite support in $Spect(\zeta) = \{0, 1, 2, ...\}$. Completion of K_0 in the norm

$$\| F \| = \sqrt{tr(F^* F)} \tag{17}$$

forms the Hilbert space K of the square integrable functions in the linear space H with the scalar product

$$(F, G) = tr(F^* G). \tag{18}$$

The formula

$$T(g)F(z) = F(gz) \tag{19}$$

defines the representation of $E(2)$ in K. (2) and (10) and the independence of the trace from the basis over which it is taken imply that the representation is unitary. Using (5) we can rewrite (19) in the form

$$T(g)F(z) = U(g)F(z)U^*(g). \tag{20}$$

Now we consider the infinitesimal form of (19). Let $g = g(re^{i\psi}, \phi)$ in (3). With the one parameter subgroups $g_1 = g(\epsilon, 0)$, $g_2 = g(i\epsilon, 0)$ and $g_3 = g(0, \epsilon)$ of $E(2)$ we associate the linear operators $K_0 \to K$

$$p_k(F) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}(T(g_k)F - F) \tag{21}$$

with the limit taken in the strong operator topology. Inserting (20) into (21) we get

$$p(F) = 2[F, z^*], \quad \overline{p}(F) = 2[z, F], \quad h(F) = [\zeta, F], \tag{22}$$

where

$$p = p_1 - ip_2, \quad \overline{p} = p_1 + ip_2, \quad h = ip_3. \tag{23}$$

For example

$$p(f(\zeta)z^n) = 2(nf(\zeta) + \zeta(f(\zeta+1) - f(\zeta)))z^{n-1}. \tag{24}$$

(18) and (22) imply the real structure in the Lie algebra of $E(2)$

$$p^* = -\overline{p}, \quad h^* = h. \tag{25}$$

The irreducible representations of $E(2)$ defined by the weight $\lambda \in R$ can be constructed in the space of square integrable functions on the circle and the matrix elements are given in terms of the Bessel functions [4]

$$t^{\lambda}_{kn}(g) = i^{n-k} e^{-i(n\phi + (k-n)\psi)} J_{n-k}(\lambda r). \tag{26}$$

Coming to our case the basis D^{λ}_k in K where the unitary irreducible representations of the group are realized will be the eigenfunctions of the complete set of commuting operators pp^* and h:

$$pp^* D^{\lambda}_k = \lambda^2 D^{\lambda}_k, \tag{27}$$

$$hD^{\lambda}_k = kD^{\lambda}_k. \tag{28}$$

The solutions of the equation (28) are

$$D^{\lambda}_k(z) = \left\{ \begin{array}{ll} z^{*k} f^{\lambda}_k(\zeta) & if \quad k \geq 0 \\ f^{\lambda}_{-k}(\zeta) z^{-k} & if \quad k \leq 0 \end{array} \right. \tag{29}$$

Inserting (28) into (27) we get

$$(k+1+\zeta) f^{\lambda}_k(\zeta+1) + (\frac{\lambda^2}{4} - 2\zeta - k - 1) f^{\lambda}_k(\zeta) + \zeta f^{\lambda}_k(\zeta-1) = 0. \tag{30}$$

By the virtue of the recurrence relation [5]

$$a\Phi(a+1, b; c) + (a-b)\Phi(a-1, b; c) + (b - 2a - c)\Phi(a, b; c) = 0 \tag{31}$$

we observe that the solutions are given in terms of the Kummer functions as

$$f^{\lambda}_k(\zeta) = \frac{(-\lambda^2)^{k/2}}{2^k k!} e^{-\frac{\lambda^2}{8}} \Phi(-\zeta, 1+k; \frac{\lambda^2}{4}). \tag{32}$$

The formula

$$L^k_n(x) = \frac{(k+n)!}{k!n!} \Phi(-n, 1+k; x) \tag{33}$$

allows us to express the basis elements in terms of the Laguerre polynomials too

$$f^{\lambda}_k(\zeta) = \frac{(-\lambda^2)^{k/2} \zeta!}{2^k (k+\zeta)!} e^{-\frac{\lambda^2}{8}} L^k_{\zeta}(\frac{\lambda^2}{4}). \tag{34}$$

The above formula is well defined since the spectrum of the operator ζ is the set of positive integers with zero. It has been well known that the Laguerre

polynomials are related to the group of complex third order triangular matrices [4]. The group parameters appears in the argument of the Laguerre polynomials. However in our case the group parameter appear in the index of this function.

To obtain the orthogonality relations we first take $z \to 1^-$ limit in the summation formula [6]

$$\sum_{n=0}^{\infty} \frac{n!}{(n+k)!} L_n^k(x) L_n^k(y) z^n = \frac{(xyz)^{-k/2}}{1-z} e^{-z\frac{x+y}{1-z}} I_k(2\frac{\sqrt{xyz}}{1-z}), \quad |z| < 1, \tag{35}$$

then use the asymptotic relation for the Bessel functions [3]

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} + \frac{e^{-x+(\nu+1/2)\pi}}{\sqrt{2\pi x}} \tag{36}$$

and employ the representation

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{\pi\epsilon}} e^{-\frac{x^2}{\epsilon}} \tag{37}$$

for the Dirac delta function. As the result we have

$$(D_k^\lambda, D_n^{\lambda'}) = \delta_{kn} \delta(\lambda^2 - \lambda'^2). \tag{38}$$

Let us make the substitution $z \to \frac{1}{\sqrt{\sigma}} z$ in (1). In $\sigma \to 0$ limit the linear space H reduces to the complex plane; and $D_k^\lambda(z)$ reduces to $t_{k0}^\lambda(g)$ which are the restriction of the matrix elements (26) on the complex plane $E(2)/U(1)$:

$$\lim_{\sigma \to 0} D_k^{\sqrt{\sigma}\lambda}(\frac{1}{\sqrt{\sigma}} r e^{i\psi}) = t_{k0}^\lambda(g). \tag{39}$$

To prove the above limit we used the asymptotic relation [5]

$$\lim_{a \to \infty} \Phi(a, b; \frac{c}{a}) = \Gamma(b) \sqrt{c^{1-b}} I_{b-1}(2\sqrt{c}). \tag{40}$$

4. The addition theorem

Let us consider the representations of $E(2)$ in the basis D_k^λ:

$$T(g) D_k^\lambda(z) = \sum_{n=-\infty}^{\infty} t_{kn}^\lambda(g) D_n^\lambda(z). \tag{41}$$

By making use of (20) the above formula can be rewritten as

$$U(g) D_k^\lambda(z) U^*(g) = \sum_{n=-\infty}^{\infty} t_{kn}^\lambda(g) D_n^\lambda(z) \tag{42}$$

which is an addition theorem useful in deriving many identities involving the Kummer and Bessel functions. For example let $g = g(r, o)$, $k \geq 0$ and $x = \lambda/2$. Then (42) reads

$$\frac{x^k}{k!} U z^{*k} \Phi(-\zeta, 1+k; x^2) U^* = \sum_{n=0}^{\infty} \frac{x^n}{n!} J_{k-n}(2rx) z^{*n} \Phi(-\zeta, 1+n; x^2) + \sum_{n=1}^{\infty} \frac{(-x)^n}{n!} J_{k+n}(2rx) \Phi(-\zeta, 1+n; x^2) z^n, \quad (43)$$

where

$$U = e^{-\frac{r^2}{2}} \sum_{n,m=0}^{\infty} \frac{(-)^m r^{n+m}}{\sqrt{n!m!}} {}_2F_0(-m, -n; -\frac{1}{r^2}) \mid m\rangle\langle n \mid . \quad (44)$$

Sandwiching (43) between the states $\langle 0 \mid$ and $\mid 0\rangle$ we get

$$\sum_{n=0}^{\infty} \frac{r^{2n}}{n!} \Phi(-n, 1+k; x^2) = k!(xr)^{-k} e^{r^2} J_k(2xr). \quad (45)$$

Multiplying (43) by U^* from the left and then sandwiching it between the states $\langle m+k \mid$ and $\mid 0\rangle$ we obtain another formula

$$\frac{(m+k)!}{m!k!} (\frac{x}{r})^k \Phi(-m, 1+k; x^2) = \sum_{n=0}^{\infty} \frac{(-xr)^n}{n!} {}_2F_0(-m-k, -n; -\frac{1}{r^2}) J_{k-n}(2xr). \quad (46)$$

It is clear that (42) can lead to many more identities, that some of them may not exist in the literature.

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